

Harmonic analysis of the functions

$$\tilde{\Delta}(x) \text{ and } N(T)$$

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Abstract

In this paper, under the Riemann hypothesis, we study the Fourier analysis about the functions $\tilde{\Delta}(x)$ and $N(T)$.

1 INTRODUCTION

Riemann hypothesis has been studied in many different ways, in this paper, we will try to use somewhat new angles to study RH. Most results of this paper are obtained under the RH. As we know, Guinand formula is a representation of $N(T)$ which is the distribution function of Riemann zeros in term of series of the prime number powers, although Guinand formula [2] is a result under the assumption of RH, it provides an explicit method to figure out all non trivial Riemann zeros. Actually, this fact is far from trivial because once we have prime number representation of $N(T)$ (Guinand formula) at hand, we can immediately restore a function via the distribution of it's zeros, so Guinand formula is equivalent to RH and we will prove it in the first section. Since Guinand formula is very important in this paper and Guinand original proof is complicated and full of the favor of harmonic analysis, we will first of all give another simple and elementary proof based on the lemma, which is the ground stone of this paper, besides, our new proof gives out a stronger conclusion than the original statement of Guinand formula. This stronger result will help us to check the truth of RH much more efficiently. In the second section we rewrite Guinand formula and Riemann-Mangoldt formula as two integral equations of two "functional variables" $\tilde{\Delta}(x)$ and $S(T)$, which seems to imply Guinand formula and Riemann-Mangoldt formula are reciprocal to each other and such integral representation will be used in the 4th section. In the third section, first of all, we derive an elementary formula based on functional equation of Riemann zeta function and lemma. This

formula provides infinitely many non trivial integral equations of $N(T)$, also, we use the elementary formula to prove a theorem which claims $|\tilde{\Delta}(x)|$ has a non-zero measurement of a positive lower bound.

2 Guinand formula with an error term and it's inverse theorem

In this section, we will give out a proof of Guinand formula with the uniformly convergent error term, besides, we also give out an inverse theorem of Guinand formula. First of all we need following notations and formulas which will be used throughout this paper[1],[3].

Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

Where the Von Mangoldt function $\Lambda(n) = \log p$ if $n = p^k$ for some k and some prime number p , $\Lambda(n) = 0$ otherwise.

Theorem 1 (Mangoldt and Riemann explicit formula)

$$\psi(x) = x - \lim_{T \rightarrow \infty} \sum_{|\text{Im} \rho| < T} \frac{x^\rho}{\rho} - \log(2\pi) + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n}$$

Where ρ runs through all non-trivial Riemann zeros.

Theorem 2 [3]

$$\psi(x) = x - \sum_{|\text{Im} \rho| < T} \frac{x^\rho}{\rho} + O\left(\frac{x \log^2 x}{T}\right)$$

we set

$$\tilde{\psi}(x) = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - x \ln(2\pi) - \sum_{n=1}^{\infty} \frac{x^{-2n+1}}{2n(2n-1)} \quad (1)$$

It's easy to prove that when x is not equal to any integer, $\tilde{\psi}(x)$ is differentialble and it's derivative is just $\psi(x)$ and it's continuous when $x > 0$ [1]. We set $\Delta(x) = \psi(x) - x$ and $\tilde{\Delta}(x) = \tilde{\psi}(x) - \frac{x^2}{2}$.

The following lemma will is important for deducing the Guinand formula with the error term.

Lemma 3 *when $s \neq \rho$,*

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) = & - \sum_{n < X} \frac{\Lambda(n)}{n^s} + \psi(X)X^{-s} + s\tilde{\psi}(X)X^{-s-1} - \\ & \frac{s(s+1)}{2(s-1)}X^{1-s} + \sum_{\rho} \frac{s(s+1)X^{\rho-s}}{\rho(\rho+1)(s-\rho)} + \sum_{n \geq 1} \frac{s(s+1)X^{-2n-2s}}{2n(2n-1)(s+2n)} \end{aligned} \quad (2)$$

Proof. Let

$$f_X(s) = \sum_{n < X} \frac{\Lambda(n)}{n^s}$$

Using integration by parts twicely, we have that

$$\begin{aligned} f_X(s) = \int_1^X x^{-s} d\psi(x) &= \psi(X)X^{-s} - \int_1^X \psi(x)dx^{-s} = \psi(X)X^{-s} + s \int_1^X \psi(x)x^{-s-1}dx \\ &= \psi(X)X^{-s} + s \int_1^X x^{-s-1} d\tilde{\psi}(x) \\ &= \psi(X)X^{-s} + s\tilde{\psi}(X)X^{-s-1} + s(s+1) \int_1^X \tilde{\psi}(x)x^{-s-2}dx \end{aligned} \quad (3)$$

and by formula 1, we can further get

$$\int_1^X \tilde{\psi}(x)x^{-s-2}dx = \int_1^X \frac{\frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - \ln(2\pi)x - \sum_{n=1}^{\infty} \frac{x^{-2n+1}}{2n(2n-1)}}{x^{s+2}} dx$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{X^{1-s}}{1-s} - \frac{1}{1-s} \right) - \sum_{\rho} \frac{1}{\rho(\rho+1)} \left(\frac{X^{\rho-s}}{\rho-s} - \frac{1}{\rho-s} \right) \\
&+ \ln(2\pi) \left(\frac{X^{-s}}{s} - \frac{1}{s} \right) + \sum_{n \geq 1} \frac{1}{2n(2n-1)} \left(\frac{X^{-s-2n}}{s+2n} - \frac{1}{s+2n} \right)
\end{aligned}$$

We collect all above terms as two groups $J_X(s)$ and $I(s)$, obviously,

$$I(s) = \frac{1}{2} \frac{1}{s-1} - \sum_{\rho} \frac{1}{\rho(\rho+1)} \frac{1}{s-\rho} - \frac{\ln(2\pi)}{s} - \sum_{n \geq 1} \frac{1}{2n(2n-1)} \frac{1}{s+2n}$$

By formula 3 and using the notations $J_X(s)$ and $I(s)$, we get

$$f_X(s) = \psi(X)X^{-s} + s\tilde{\psi}(X)X^{-s-1} + s(s+1)J_X(s) + s(s+1)I(s) \quad (4)$$

and

$$s(s+1)I(s) = \frac{s(s+1)}{2(s-1)} - \sum_{\rho} \frac{s(s+1)}{\rho(\rho+1)(s-\rho)} - (s+1)\ln(2\pi) - \sum_{n \geq 1} \frac{s(s+1)}{2n(2n-1)(s+2n)}$$

By the following identity,

$$\frac{s(s+1)}{z(z+1)(s-z)} = \frac{s}{z(z+1)} + \frac{1}{s-z} + \frac{1}{z}$$

we have that

$$s(s+1)I(s) = -\frac{\zeta'}{\zeta}(s) + as + b \quad (5)$$

where a, b are some constants which can be determined immediately. According 4 and 5, we get a new representation of $\frac{\zeta'}{\zeta}(s)$ when $s \neq \rho$ as follows:

$$\begin{aligned}
&\frac{\zeta'}{\zeta}(s) = - \sum_{n < X} \frac{\Lambda(n)}{n^s} + \psi(X)X^{-s} + s\tilde{\psi}(X)X^{-s-1} \\
&- \frac{s(s+1)}{2(s-1)}X^{1-s} + \sum_{\rho} \frac{s(s+1)X^{\rho-s}}{\rho(\rho+1)(s-\rho)} + \sum_{n \geq 1} \frac{s(s+1)X^{-2n-s}}{2n(2n-1)(s+2n)} + as + b \quad (6)
\end{aligned}$$

Since when $Res > 1$,

$$\frac{\zeta'}{\zeta}(s) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

Let $X \rightarrow \infty$ on the right side of 6 when $Res > 1$, we immediately get $a = b = 0$, that follows our theorem.

As we know,

$$\log \zeta(s_0) = \int_2^{s_0} \frac{\zeta'}{\zeta}(s) ds$$

,where the integral path is a positive orient half rectangle with vertices $2, 2 + iT, \sigma + iT$ and $s_0 = \sigma + iT$ $s_0 \neq \rho$. Taking this complex integral on both sides of 6, we directly get following theorem:

Theorem 4 *When $s_0 \neq \rho$, we have that*

$$\begin{aligned} \log \zeta(s_0) = \sum_{n < X} \frac{\Lambda(n)}{(\log n) n^{s_0}} - \frac{\Delta(X)}{(\log X) X^{s_0}} - \frac{\tilde{\Delta}(X)}{(\log X) X^{s_0+1}} \left(s_0 + \frac{1}{\ln X} \right) \\ + \int_2^{s_0} \frac{X^{1-s}}{1-s} ds + \tilde{J}_X(s_0) + C_0 \end{aligned} \quad (7)$$

Where

$$\begin{aligned} \tilde{J}_X(s_0) = \int_2^{s_0} s(s+1)J(X)ds = -\frac{1}{\ln X} \sum_{\rho} \frac{1}{\rho(\rho+1)} \left[\frac{s_0(s_0+1)X^{\rho-s_0}}{s_0-\rho} \right. \\ \left. - \int_2^{s_0} X^{\rho-s} \left(\frac{2s+1}{s-\rho} - \frac{s^2+s}{(s-\rho)^2} \right) ds \right] \\ - \frac{1}{\ln X} \sum_{n \geq 1} \frac{1}{2n(2n-1)} \left[\frac{s_0(s_0+1)X^{-2n-s}}{s+2n} - \int_2^{s_0} X^{-2n-s} \left(\frac{2s+1}{s+2n} - \frac{s^2+s}{(s+2n)^2} \right) ds \right] \end{aligned}$$

and C_0 is a real constant.

Proof. Using integration by parts and collecting all terms containing X^{1-s} , we immediately get above results. setting $s_0 = \frac{1}{2} + iT$ in the formula 7 and taking imaginary parts on both sides, we have that

Theorem 5 *If the Riemann hypothesis is true, and δ is the distance between T and the coordinate of the nearest Riemann zero, we have*

$$\pi S(T) = - \sum_{n < X} \frac{\Lambda(n) \sin(T \log n)}{\sqrt{n}} + \frac{\Delta(X) \sin(T \log X)}{\sqrt{X}(\log X)} + \operatorname{Im} \left(\int_2^{\frac{1}{2}+iT} \frac{X^{1-s}}{1-s} ds \right) + O\left(\frac{T^3}{\delta^2 \ln X}\right) \quad (8)$$

in the limit language, we have

$$\begin{aligned} \pi S(T) = & -\lim_{X \rightarrow \infty} \left[\sum_{n < X} \frac{\Lambda(n) \sin(T \log n)}{\sqrt{n}(\log n)} \right. \\ & \left. - \frac{\Delta(X) \sin(T \log X)}{\sqrt{X}(\log X)} - \operatorname{Im} \left(\int_2^{\frac{1}{2}+iT} \frac{X^{1-s}}{1-s} ds \right) \right] \end{aligned} \quad (9)$$

From now on, we will prove formula 9 is the same as Guinand formula. To achieve it, we need to make some simplification as follows:

Let's first simplify the term

$$\operatorname{Im} \left(\int_2^{\frac{1}{2}+iT} \frac{X^{1-s}}{1-s} ds \right)$$

, Let's transform the original integral path which is half rectangle with vertices $2, 2+iT, \frac{1}{2}+iT$ to another half rectangle with vertices $2, \frac{1}{2}, \frac{1}{2}+iT$ and orient is clockwise, we get

$$\int_2^{\frac{1}{2}+iT} \frac{X^{1-s}}{1-s} ds = i \int_0^T \frac{X^{\frac{1}{2}-it}}{\frac{1}{2}-it} dt + \int_{\Gamma_r} \frac{X^{1-s}}{1-s} ds \quad (10)$$

Where $\gamma_r = [\frac{1}{2}, 1-r] \cup S_r \cup [1+r, 2]$ and S_r is upper half semi-circle with radius r and centered at $z = 1$ Taking imaginary part on both side of 10, we get the first term of right hand side is equal to

$$\sqrt{X} \int_0^T \frac{2 \cos(t \log X) + 4t \sin(t \log X)}{1 + 4t^2} dt$$

which is set to be $f_1(T, X)$ For the second term of right side of 10, we have that

$$\begin{aligned} \operatorname{Im}\left(\int_{\Gamma_r} \frac{X^{1-s}}{1-s} ds\right) &= \operatorname{Im}\left(\int_{J_r} \frac{X^{1-s}}{1-s} ds\right) \\ &= \lim_{r \rightarrow 0} \operatorname{Im}\left(\int_{J_r} \frac{X^{1-s}}{1-s} ds\right) = -\pi \end{aligned} \quad (11)$$

Let's pick up the second term of right side of 44 i.e. $\int_1^X \frac{\sin(T \log t)}{\sqrt{t \log t}} dt$ and set it to be $f_2(T, X)$, Since $\frac{\sin(T \log t)}{\sqrt{t \log t}}$ is continuously differentiable with respect to T , we have that

$$\begin{aligned} \frac{\partial f_2}{\partial T} &= \int_1^X \frac{\cos(T \log t)}{\sqrt{t}} dt \\ &= \int_0^{\ln X} e^{\frac{1}{2}u} \cos(Tu) du = \frac{\sqrt{X}}{1+4T^2} [2\cos(T \log X) + 4T \sin(T \log X)] - \frac{2}{1+4T^2} \end{aligned} \quad (12)$$

in which we have used the substitution $u = \log t$

We can also notice that

$$\frac{\partial f_1}{\partial T} = \frac{\sqrt{X}}{1+4T^2} [2\cos(T \log X) + 4T \sin(T \log X)]$$

By 12, we get

$$\frac{\partial f_1}{\partial T} - \frac{\partial f_2}{\partial T} = \frac{2}{1+4T^2}$$

Thus,

$$\begin{aligned} f_1(T, X) - f_2(T, X) &= \int_0^T \frac{2}{1+4t^2} dt \\ &= \arctan 2T \end{aligned} \quad (13)$$

Consequently, by 10, 11, 13

$$\operatorname{Im}\left(\int^{\frac{1}{2}+iT} \frac{X^{1-s}}{1-s} ds\right) - \int_1^X \frac{\sin(T \log t)}{\sqrt{t \log t}} dt = \arctan 2T - \pi \quad (14)$$

Let's single out the term

$$\frac{\sin(T \log X)}{\log X} \left\{ \sum_{n < X} \Lambda(n) n^{-\frac{1}{2}} - 2X^{\frac{1}{2}} \right\}$$

in right hand of 44 and get it simplified as follows:

$$\begin{aligned} & \frac{\sin(T \log X)}{\log X} \left\{ \sum_{n < X} \Lambda(n) n^{-\frac{1}{2}} - 2X^{\frac{1}{2}} \right\} \\ &= \frac{\sin(T \log X)}{\log X} \left[\int_1^X x^{-\frac{1}{2}} d\psi(x) - 2\sqrt{X} \right] \\ &= \frac{\sin(T \log X)}{\log X} \left[\psi(X) X^{-\frac{1}{2}} + \frac{1}{2} \int_1^X \psi(x) x^{-\frac{3}{2}} dx - 2\sqrt{X} \right] \\ &= \frac{\Delta(X) \sin(T \log X)}{\sqrt{X}(\log X)} + \frac{\sin(T \log X)}{2 \log X} \int_1^X \Delta(x) x^{-\frac{3}{2}} dx \end{aligned} \quad (15)$$

and by the theorem 2 , we have that

$$\begin{aligned} \int_1^X \Delta(x) x^{-\frac{3}{2}} dx &= - \sum_{\rho} \frac{X^{\rho-\frac{1}{2}} - 1}{\rho(\rho - \frac{1}{2})} + 2 \ln(2\pi)(X^{-\frac{1}{2}} - 1) \\ &\quad - \sum_{n \geq 1} \frac{X^{-2n-\frac{1}{2}} - 1}{2n(2n + \frac{1}{2})} = O(1) \end{aligned}$$

With formula 15, we have

$$\frac{\sin(T \log X)}{\log X} \left[\sum_{n \leq X} \Lambda(n) n^{-\frac{1}{2}} - 2X^{\frac{1}{2}} \right] = \frac{\Delta(X) \sin(T \log X)}{\sqrt{X}(\log X)} + O\left(\frac{1}{\log X}\right) \quad (16)$$

Since

$$\begin{aligned} N(T) &= \frac{1}{\pi} \arg \xi\left(\frac{1}{2} + iT\right) \\ &= \frac{1}{\pi} \arg s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \Big|_{s=\frac{1}{2}+iT} \\ &= \frac{1}{\pi} \arg\left(-\frac{1}{4} - T^2\right) - \frac{T \ln \pi}{2\pi} + \frac{1}{\pi} \arg \Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) + S(T) \\ &= 1 - \frac{T \ln \pi}{2\pi} + \frac{1}{\pi} \arg \Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) + S(T) \end{aligned} \quad (17)$$

Whenever T is not equal to any coordinates of some Riemann zeros, we can rewrite Guinand formula 44 as follows

$$\pi S(T) = F_X(T) + \arctan(2T) - \pi + \frac{1}{2} \arg \Gamma\left(\frac{1}{2} + iT\right) - \arg \Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) - \frac{T \ln 2}{2} - \frac{1}{4} \arctan(\sinh \pi T) \quad (18)$$

Where

$$F_X(T) = -\lim_{X \rightarrow \infty} \left[\sum_{n \leq X} \Lambda(n) \frac{\sin(T \log n)}{\sqrt{n} \log n} - \int_1^X \frac{\sin(T \log t)}{\sqrt{t} \log t} dt - \frac{\sin(T \log X)}{\log X} \left[\sum_{n \leq X} \Lambda(n) n^{-\frac{1}{2}} - 2X^{\frac{1}{2}} \right] \right]$$

Using equation 18 (Guinand formula) minus equation 9 and notice 14 and 16, we get that

$$0 = \frac{1}{2} \arg \Gamma\left(\frac{1}{2} + iT\right) - \arg \Gamma\left(\frac{1}{4} + \frac{iT}{2}\right) - \frac{T \ln 2}{2} - \frac{1}{4} \arctan(\sinh \pi T) \quad (19)$$

When T is not coordinates of some Riemann zeros. We set right side of 19 to be $d(T)$, then we just need to prove that $d(T) \equiv 0$ when $T > 0$. Let's show it as follows: By rewriting

$$\frac{T \ln 2}{2} = \arg 2^{\frac{iT}{2}}$$

and

$$\arctan(\sinh \pi T) = \arg(1 + i \sinh \pi T)$$

, we have that

$$4d(T) = \arg \frac{\Gamma^2\left(\frac{1}{2} + iT\right)}{\Gamma^4\left(\frac{1}{4} + \frac{iT}{2}\right) 4^{iT} (1 + i \sinh \pi T)}$$

$$\begin{aligned}
&= \text{Im} \log \frac{\Gamma^2(\frac{1}{2} + iT)}{\Gamma^4(\frac{1}{4} + \frac{iT}{2}) 4^{iT} (1 + i \sinh \pi T)} \\
&= \frac{1}{2i} \left[\log \frac{\Gamma^2(\frac{1}{2} + iT)}{\Gamma^4(\frac{1}{4} + \frac{iT}{2}) 4^{iT} (1 + i \sinh \pi T)} - \log \frac{\Gamma^2(\frac{1}{2} - iT)}{\Gamma^4(\frac{1}{4} - \frac{iT}{2}) 4^{-iT} (1 - i \sinh \pi T)} \right]
\end{aligned}$$

Let $s = iT$, then

$$\sinh \pi T = -i \sin \pi s$$

and

$$4d(T) = \frac{1}{2i} \log \frac{\Gamma^2(\frac{1}{2} + s) \Gamma^4(\frac{1}{4} - \frac{s}{2}) (1 - \sin \pi s)}{\Gamma^2(\frac{1}{2} - s) \Gamma^4(\frac{1}{4} + \frac{s}{2}) 4^{2s} (1 + \sin \pi s)} \quad (20)$$

Let

$$g(s) = \frac{\Gamma^2(\frac{1}{2} + s) \Gamma^4(\frac{1}{4} - \frac{s}{2}) (1 - \sin \pi s)}{\Gamma^2(\frac{1}{2} - s) \Gamma^4(\frac{1}{4} + \frac{s}{2}) 4^{2s} (1 + \sin \pi s)}$$

then $g(s)$ is a meromorphic function in the whole complex number plane and by the formula 20, $g(s)|_{s=iT} = 1$ when $T > 0$. For the convenience of factorizing $g(s)$, let's set $s = \frac{1}{2} - z$ and reset $f(z) = g(s)$ we have that

$$f(z) = \frac{\Gamma^2(1 - z) \Gamma^4(\frac{1}{2} z) \sin^2(\frac{\pi}{2} z)}{\Gamma^2(z) \Gamma^4(\frac{1}{2} - \frac{1}{2} z) 4^{1-2z} \cos^2(\frac{\pi}{2} z)} \quad (21)$$

We just need to prove that $f(z) \equiv 1$ for any $z \in C$, that can be derived by the formula

$$\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}$$

With the formulas 16,18,19, we can rewrite the formula 8 as:

$$\begin{aligned}
\pi S(T) = & - \sum_{n < X} \frac{\Lambda(n) \sin(T \log n)}{\sqrt{n}} + \frac{\Delta(X) \sin(T \log X)}{\sqrt{X}(\log X)} + \int_1^X \frac{\sin(T \log y)}{\sqrt{y} \log y} dy \\
& + \arctan(2T) - \pi + O\left(\frac{T^3}{\delta^2 \ln X}\right)
\end{aligned}$$

Furthermore, we can rewrite above formula as an integral equation, first of all,

$$\begin{aligned}
\sum_{n < X} \frac{\Lambda(n) \sin(T \log n)}{\sqrt{n}} &= \int_a^X \frac{\sin(T \log y)}{\sqrt{y} \log y} d\psi(y) \\
&= \int_a^X \frac{\sin(T \log y)}{\sqrt{y} \log y} dy + \int_a^X \frac{\sin(T \log y)}{\sqrt{y} \log y} d\Delta(y) \\
&= \int_a^X \frac{\sin(T \log y)}{\sqrt{y} \log y} dy + \frac{\Delta(X) \sin(T \log X)}{\sqrt{X} (\log X)} - \frac{\Delta(a) \sin(T \log a)}{\sqrt{a} (\log a)} \\
&\quad - \int_a^X \frac{T \cos(T \ln y) - \sin(T \ln y) (\frac{\ln y}{2} + 1)}{y \sqrt{y} \ln^2 y} \Delta(y) dy
\end{aligned}$$

Where $1 < a < 2$.

We substitute above formula into 2, we get that

$$\begin{aligned}
S(T) &= -\frac{1}{\pi} \int_a^X \frac{T \cos(T \ln y) - \sin(T \ln y) (\frac{\ln y}{2} + 1)}{y \sqrt{y} \ln^2 y} \Delta(y) dy \\
&\quad - \frac{1}{\pi} \left[\int_a^1 \frac{\sin(T \log y)}{\sqrt{y} \log y} dy - \frac{\Delta(a) \sin(T \log a)}{\sqrt{a} (\log a)} + \arctan(2T) - \pi \right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\pi} \int_a^X \frac{T \cos(T \ln y) - \sin(T \ln y) (\frac{\ln y}{2} + 1)}{y \sqrt{y} \ln^2 y} \Delta(y) dy \\
&= \frac{1}{\pi} \int_a^X \frac{T \cos(T \ln y) - \sin(T \ln y) (\frac{\ln y}{2} + 1)}{y \sqrt{y} \ln^2 y} d\tilde{\Delta}(y) \\
&= \tilde{\Delta}(X) \frac{T \cos(T \ln X) - \sin(T \ln X) (\frac{\ln X}{2} + 1)}{X \sqrt{X} \ln^2 X} - \tilde{\Delta}(a) \frac{T \cos(T \ln a) - \sin(T \ln a) (\frac{\ln a}{2} + 1)}{a \sqrt{a} \ln^2 a} \\
&\quad - \int_a^X \tilde{\Delta}(y) d \frac{T \cos(T \ln y) - \sin(T \ln y) (\frac{\ln y}{2} + 1)}{y \sqrt{y} \ln^2 y}
\end{aligned}$$

and

$$\int_a^X \tilde{\Delta}(y) d \frac{T \cos(T \ln y) - \sin(T \ln y) \left(\frac{\ln y}{2} + 1 \right)}{y \sqrt{y} \ln^2 y} = \int_a^X F(T, y) \tilde{\Delta}(y) dy \quad (22)$$

Where

$$\begin{aligned} F(T, y) = & -\frac{T^2 \sin(T \ln y)}{y^{\frac{5}{2}} \ln y} - \frac{2T \cos(T \ln y)}{y^{\frac{5}{2}} \ln y} + \frac{3 \sin(T \ln y)}{4y^{\frac{5}{2}} \ln y} \\ & - \frac{2T \cos(T \ln y)}{y^{\frac{5}{2}} \ln^2 y} + \frac{2 \sin(T \ln y)}{y^{\frac{5}{2}} \ln^2 y} + \frac{2 \sin(T \ln y)}{y^{\frac{5}{2}} \ln^3 y} \end{aligned}$$

By 2,2,22 and when t is not the coordinate of a Riemann zero, let $X \rightarrow \infty$, we have following integral equation

$$S(t) = -\frac{1}{\pi} \int_a^\infty F(t, y) \tilde{\Delta}(y) dy + g(a, t) \quad (23)$$

Where

$$\begin{aligned} g(a, t) = & -\frac{1}{\pi} \left[\int_a^1 \frac{\sin(t \log y)}{\sqrt{y} \log y} dy - \frac{\Delta(a) \sin(t \log a)}{\sqrt{a} (\log a)} \right. \\ & \left. + \tilde{\Delta}(a) \frac{t \cos(t \ln a) - \sin(t \ln a) \left(\frac{\ln a}{2} + 1 \right)}{a \sqrt{a} \ln^2 a} + \arctan(2t) - \pi \right] \end{aligned}$$

, $1 < a < 2$

3 Representing $\tilde{\Delta}(x)$ in term of $S(T)$

In this section, under the RH , we will represent $\tilde{\Delta}(x)$ as an integral of $S(T)$ via Riemann-Von Mangoldt formula. Let $N(T)$ be a function counting

the number of non-trivial Riemann zeros whose imaginary is between 0 and T , under the RH, we can rewrite the formula in term of $N(T)$ as follows,

$$\tilde{\Delta}(x) = -\sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho+1)} - x \ln(2\pi) - \sum_{n=1}^{\infty} \frac{x^{-2n+1}}{2n(2n-1)} \quad (24)$$

$$= -\int_0^{\infty} \left[\frac{x^{\frac{3}{2}+it}}{(\frac{1}{2}+it)(\frac{3}{2}+it)} + \frac{x^{\frac{3}{2}-it}}{(\frac{1}{2}-it)(\frac{3}{2}-it)} \right] dN(t) + f(x) \quad (25)$$

$$= -2x^{\frac{3}{2}} \int_0^{\infty} \frac{(\frac{3}{4}-t^2)\cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4}-t^2)^2} dN(t) + f(x) \quad (26)$$

Where

$$f(x) = -x \ln(2\pi) - \sum_{n=1}^{\infty} \frac{x^{-2n+1}}{2n(2n-1)}$$

Noticing 17, set

$$g(t) = 1 - \frac{t \ln \pi}{2\pi} + \frac{1}{\pi} \arg \Gamma\left(\frac{1}{4} + \frac{it}{2}\right)$$

Thus we have

$$\tilde{\Delta}(x) = -2x^{\frac{3}{2}} \int_0^{\infty} \frac{(\frac{3}{4}-t^2)\cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4}-t^2)^2} dS(t) \quad (27)$$

$$-2x^{\frac{3}{2}} \int_0^{\infty} \frac{(\frac{3}{4}-t^2)\cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4}-t^2)^2} dg(t) + f(x) \quad (28)$$

Putting the last two terms together and setting it to be $\tilde{f}(x)$, we get that

$$\tilde{\Delta}(x) = -2x^{\frac{3}{2}} \int_0^{\infty} \frac{(\frac{3}{4}-t^2)\cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4}-t^2)^2} dS(t) + \tilde{f}(x) \quad (29)$$

Using integration by parts and noticing $S(t) = O(\log t)$, we have

$$\begin{aligned} & \int_0^{\infty} \frac{(\frac{3}{4}-t^2)\cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4}-t^2)^2} dS(t) \\ &= -\frac{4S(0)}{3} - \int_0^{\infty} S(t) d \frac{(\frac{3}{4}-t^2)\cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4}-t^2)^2} \end{aligned} \quad (30)$$

From now on, we are going to evaluate the second term of 30 in detail for the convenience of checking.

$$\begin{aligned}
& \int_0^\infty S(t) d \frac{(\frac{3}{4} - t^2) \cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4} - t^2)^2} \\
&= \int_0^\infty S(t) \frac{[(\frac{3}{4} - t^2) \cos(t \ln x) + 2t \sin(t \ln x)]' [4t^2 + (\frac{3}{4} - t^2)^2]}{[4t^2 + (\frac{3}{4} - t^2)^2]^2} dt \\
&\quad - \int_0^\infty S(t) \frac{[(\frac{3}{4} - t^2) \cos(t \ln x) + 2t \sin(t \ln x)] [4t^2 + (\frac{3}{4} - t^2)^2]'}{[4t^2 + (\frac{3}{4} - t^2)^2]^2} dt \\
&= \int_0^\infty S(t) \frac{[-2t \cos(t \ln x) - (\frac{3}{4} \ln x) \sin(t \ln x) + t^2 \ln x \sin(t \ln x) + 2 \sin(t \ln x) + 2t \ln x \cos(t \ln x)] [t^4 + \frac{5}{2}]}{[4t^2 + (\frac{3}{4} - t^2)^2]^2} dt \\
&\quad - \int_0^\infty S(t) \frac{[\frac{3}{4} \cos(t \ln x) - t^2 \cos(t \ln x) + 2t \sin(t \ln x)] [4t^3 + 5t]}{[4t^2 + (\frac{3}{4} - t^2)^2]^2} dt \\
&= \int_0^\infty S(t) \frac{-2t^5 \cos(t \ln x) - \frac{3}{4} t^4 \sin(t \ln x) + t^6 \ln x \sin(t \ln x) + 2t^4 \sin(t \ln x) + 2t^5 \ln x \cos(t \ln x)}{(t^4 + \frac{5}{2} t^2 + \frac{9}{16})^2} dt \\
&\quad + \int_0^\infty S(t) \frac{-5t^3 \cos(t \ln x) - \frac{15}{8} t^2 \sin(t \ln x) + \frac{5}{2} t^4 \ln x \sin(t \ln x) + 5t^2 \sin(t \ln x) + 5t^3 \ln x \cos(t \ln x)}{(t^4 + \frac{5}{2} t^2 + \frac{9}{16})^2} dt \\
&\quad + \int_0^\infty S(t) \frac{-\frac{9}{8} t \cos(t \ln x) - (\frac{27}{64} \ln x) \sin(t \ln x) + \frac{9}{16} t^2 \ln x \sin(t \ln x) + \frac{9}{8} \sin(t \ln x) + \frac{9}{8} t \ln x \cos(t \ln x)}{(t^4 + \frac{5}{2} t^2 + \frac{9}{16})^2} dt \\
&\quad - \int_0^\infty S(t) \frac{3t^3 - 4t^5 \cos(t \ln x) + 8t^4 \sin(t \ln x) + \frac{15}{4} t \cos(t \ln x) - 5t^3 \cos(t \ln x) + 10t^2 \sin(t \ln x)}{(t^4 + \frac{5}{2} t^2 + \frac{9}{16})^2} dt
\end{aligned}$$

To summarize, we get

$$\tilde{\Delta}(x) = -2x^{\frac{3}{2}} \int_0^\infty K(x, t) S(t) dt + f(x) \tag{31}$$

Where we still let

$$f(x) = -2x^{\frac{3}{2}} \int_0^\infty \frac{(\frac{3}{4} - t^2) \cos(t \ln x) + 2t \sin(t \ln x)}{4t^2 + (\frac{3}{4} - t^2)^2} dg(t) - x \ln(2\pi) - \sum_{n=1}^\infty \frac{x^{-2n+1}}{2n(2n-1)} + \frac{8S(0)}{3} x^{\frac{3}{2}}$$

and where

$$\begin{aligned} K(x, t) = & \ln x [t^6 \sin(t \ln x) + 2t^5 \cos(t \ln x) + \frac{7}{4} t^4 \sin(t \ln x) + 5t^3 \cos(t \ln x) - \frac{21}{16} t^2 \sin(t \ln x) \\ & + \frac{9}{8} t \cos(t \ln x) - \frac{27}{64} \sin(t \ln x)] + 2t^5 \cos(t \ln x) - 6t^4 \sin(t \ln x) - 3t^3 \cos(t \ln x) - 5t^2 \sin(t \ln x) \\ & - \frac{39}{8} t \cos(t \ln x) + \frac{9}{8} \sin(t \ln x) \end{aligned}$$

By the equations 23,31, we can get a system of integral equations

$$\begin{cases} \tilde{\Delta}(x) = -2x^{\frac{3}{2}} \int_0^\infty K(x, t) S(t) dt + f(x) \\ S(t) = -\frac{1}{\pi} \int_a^\infty F(t, y) \tilde{\Delta}(y) dy + g(a, t) \end{cases} \quad (32)$$

We shall prove an inverse theorem of the Guinnand formula as follows.

Theorem 6 *Let $f(t)$ be a function which is continous on the interval $[0, +\infty]$ except some discrete points, which forms a set E , and $f(t) = \lim_{X \rightarrow \infty} f_X(t)$,*

$$f_X(t) = - \sum_{n < X} \frac{\Lambda(n) \sin(T \log n)}{\sqrt{n}} + \frac{\Delta(X) \sin(T \log X)}{\sqrt{X} (\log X)} + \int_1^X \frac{\sin(T \log y)}{\sqrt{y} \log y} dy$$

and $f_X(t) = O(\log t)$ on $[0, +\infty] \setminus E$, then the RH holds

Proof. Considering the function

$$F(s) = \int_0^{+\infty} \frac{2(s-1)}{(s-\frac{1}{2})^2 + t^2} df(t)$$

and

$$f_X(s) = \int_0^{+\infty} \frac{2(s-1)}{(s-\frac{1}{2})^2 + t^2} df_X(t)$$

, where $Res > \frac{1}{2}$ Using integration by parts,

$$F(s) = -\frac{2f(0)}{s-\frac{1}{2}} + \int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} f(t) dt$$

and

$$f_X(s) = -\frac{2f_X(0)}{s-\frac{1}{2}} + \int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} f_X(t) dt$$

Since $f(t) = \lim_{X \rightarrow \infty} f_X(t)$ and $f_X(t) = O(\log t)$ on $[0, +\infty] \setminus E$, by the Lebesgue CCL, $\lim_{X \rightarrow \infty} f_X(s) = F(s)$ and

$$\int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} f_X(t) dt = \int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} \left[-\sum_{n < X} \frac{\Lambda(n) \sin(t \log n)}{\sqrt{n}} + \frac{\Delta(X) \sin(t \log X)}{\sqrt{X}(\log X)} + J \right] dt$$

Using residue theorem, we have that

$$\int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} \frac{\Lambda(n) \sin(t \log n)}{\sqrt{n}} dt = \frac{\Lambda(n)}{n^s}$$

Similarly,

$$\int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} \left[\frac{\Delta(X) \sin(t \log X)}{\sqrt{X}(\log X)} \right] dt = \frac{\Delta(X)}{X^{s+\frac{1}{2}} \log X}$$

and

$$\int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} \left[\int_1^X \frac{\sin(T \log y)}{\sqrt{y} \log y} dy \right] dt = \frac{1}{1-s} [X^{1-s} - 1]$$

To summarize,

$$\int_0^{+\infty} \frac{4(s-1)t}{[(s-\frac{1}{2})^2 + t^2]^2} f_X(t) dt = -\sum_{n < X} \frac{\Lambda(n)}{n^s} + \frac{\Delta(X)}{X^{s+\frac{1}{2}} \log X} + \frac{1}{1-s} [X^{1-s} - 1] \quad (34)$$

Since for any $Res > \frac{1}{2}$, $\lim_{X \rightarrow \infty} f_X(s) = F(s)$, and $F(s)$ is analytical in the half plane $Res > \frac{1}{2}$, we notice that when $Res > 1$, we have

$$\lim_{X \rightarrow \infty} f_X(s) = - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} + \frac{1}{s-1} \quad (35)$$

$$= - \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = F(s) \quad (36)$$

Which means that RH is true.

4 Lower bound of $\tilde{\Delta}(x)$

First of all , Let's derive a formula based on the functional equation and formula, since

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -s(s+1) \int_1^X \tilde{\Delta}(x) x^{-s-2} dx \\ &- \frac{s(s+1)}{2(s-1)} X^{1-s} + \sum_{\rho} \frac{s(s+1)X^{\rho-s}}{\rho(\rho+1)(s-\rho)} + \sum_{n \geq 1} \frac{s(s+1)X^{-2n-s}}{2n(2n-1)(s+2n)} \end{aligned} \quad (37)$$

As we know, by the functional equation and 37, we have

$$Re \frac{\zeta'(s)}{\zeta(s)} \Big|_{s=\frac{1}{2}+it} = \frac{\ln \pi}{2} - \frac{1}{2} Re \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} + it \right) \quad (38)$$

Evaluating real part at $s = \frac{1}{2} + it$ on both sides of 39, we get that

$$\begin{aligned} \int_1^X \frac{\tilde{\Delta}(x)}{x^{\frac{5}{2}}} [(\frac{3}{4} - t^2) \cos(t \ln x) + 2t \sin(t \ln x)] dx &- \sum_{t_{\rho}} \frac{(\frac{3}{4} - t^2)(\frac{3}{4} - t_{\rho}^2) + 4tt_{\rho}}{(\frac{3}{4} - t_{\rho}^2)^2 + 4t_{\rho}^2} \frac{\sin(t_{\rho} - t) \ln X}{t_{\rho} - t} \\ &- \sum_{t_{\rho}} \frac{(\frac{3}{4} - t^2)(\frac{3}{4} - t_{\rho}^2) - 4tt_{\rho}}{(\frac{3}{4} - t_{\rho}^2)^2 + 4t_{\rho}^2} \frac{\sin(t_{\rho} + t) \ln X}{t_{\rho} + t} \end{aligned}$$

$$\begin{aligned}
& + \sum_{t_\rho} \frac{\frac{3}{2} + 2tt_\rho}{(\frac{3}{4} - t_\rho^2)^2 + 4t_\rho^2} \cos(t_\rho - t) \ln X \\
& + \sum_{t_\rho} \frac{\frac{3}{2} - 2tt_\rho}{(\frac{3}{4} - t_\rho^2)^2 + 4t_\rho^2} \cos(t_\rho + t) \ln X = g_X(t) \quad (39)
\end{aligned}$$

Where

$$g_X(t) = \operatorname{Re} \left[-\frac{\zeta'(s)}{\zeta(s)} + \sum_{n \geq 1} \frac{s(s+1)X^{-2n-s}}{2n(2n-1)(s+2n)} \right]_{s=\frac{1}{2}+it}$$

By above formula, when $t \neq t_\rho$,

$$\int_1^X \frac{\tilde{\Delta}(x)}{x^{\frac{5}{2}}} \left[\left(\frac{3}{4} - t^2 \right) \cos(t \ln x) + 2t \sin(t \ln x) \right] dx = O(1) \quad (40)$$

otherwise,

$$\int_1^X \frac{\tilde{\Delta}(x)}{x^{\frac{5}{2}}} \left[\left(\frac{3}{4} - t_\rho^2 \right) \cos(t_\rho \ln x) + 2t_\rho \sin(t_\rho \ln x) \right] dx = \ln X + O(1) \quad (41)$$

Where $\rho = \frac{1}{2} + it_\rho$ are Riemann zeros, to simplify LHS of 41, set $\theta_\rho = \arctan \frac{\frac{3}{4} - t_\rho^2}{2t_\rho}$, then we have

$$\int_1^X \frac{\tilde{\Delta}(x)}{x^{\frac{5}{2}}} \sin(t_\rho \ln x + \theta_\rho) dx = \frac{1}{\sqrt{(\frac{3}{4} - t_\rho^2)^2 + 4t_\rho^2}} \ln X + O(1) \quad (42)$$

Let $u = \ln x$, above formula can be reduced to

$$\int_1^u \frac{\tilde{\Delta}(e^y)}{e^{\frac{5}{2}y}} \sin(t_\rho y + \theta_\rho) dy = \frac{1}{\sqrt{(\frac{3}{4} - t_\rho^2)^2 + 4t_\rho^2}} u + R_\rho(u)$$

Set $f_\rho(u) = \frac{\tilde{\Delta}(e^u)}{e^{\frac{5}{2}u}} \sin(t_\rho u + \theta_\rho)$, and $A_\rho = \frac{1}{\sqrt{(\frac{3}{4} - t_\rho^2)^2 + 4t_\rho^2}}$ Thus

We have simplified form

$$\int_0^X f_\rho(t) dt = A_\rho X + R_\rho(X)$$

Let $g(t) = \frac{\tilde{\Delta}(e^t)}{e^{\frac{3}{2}t}}$ and $\max_{0 \leq t < +\infty} |g(t)| = C_1$, $\mu(x) = m\{t \mid |g(t)| \leq x\}$,
 $\bar{\mu}(x) = X - \mu(x)$, and $E_x = \{t \mid |g(t)| \leq x\}$

By choosing any $x < C_1$, we have following estimate

$$\begin{aligned} A_\rho X + R_\rho(X) &= \int_0^X f_\rho(t) dt \leq \int_0^X |g(t)| dt = \int_{E_x} |g(t)| dt + \int_{[0, X] \setminus E_x} |g(t)| dt \\ &\leq x\mu(x) + C_1(X - \mu(x)) \end{aligned}$$

When X is big enough, we have

$$\mu(x) \leq \frac{C_1 - A_\rho}{C_1 - x} X + \frac{C_{0\rho}}{C_1 - x}$$

or

$$\bar{\mu}(x) \geq \frac{A_\rho - x}{C_1 - x} X - \frac{C_{0\rho}}{C_1 - x}$$

Where $C_{0\rho} = \max_{0 < t < \infty} |R_\rho(t)|$ and set $F_x^X = \{u \mid |\frac{\tilde{\Delta}(u)}{u^{\frac{3}{2}}}| > x, 0 < u < X\}$

Therefore $m(F_x^X) \geq \frac{A_\rho - x}{C_1 - x} X - \frac{C_{0\rho}}{C_1 - x}$

Where $\tilde{\Delta}(u) = \sum_{n \leq u} (n - \psi(n)) \Lambda(n) - \frac{u^2}{2}$

Choosing $\rho = \rho_0 = \frac{1}{2} + 14.134\dots$ which is the first non-trivial Riemann zero, we have following theorem

Theorem 7 When X is big enough, $m(F_x^X) \geq \frac{A_{\rho_0} - x}{C_1 - x} X - \frac{C_{0\rho_0}}{C_1 - x}$

Set

$$F_X(t) = \int_1^X \frac{\tilde{\Delta}(x)}{x^{\frac{5}{2}}} [(\frac{3}{4} - t^2) \cos(t \ln x) + 2t \sin(t \ln x)] dx$$

, and $G_X(t) = \int_0^t F_X(y) dy$, by the formula 40,42, and Guinand formula i.e $\lim_{X \rightarrow \infty} G_X(t) = N(t)$, we can conjecture that when $X \rightarrow \infty$, $F_X(t)$ will

behave like a distribution more than an ordinary function. Let's verify it as follows:

First of all, we notice that

$$\begin{aligned} & \int_0^{+\infty} e^{-\epsilon u} \left[\left(\frac{3}{4} - k^2 \right) \cos(ku) + 2k \sin(ku) \right] \frac{\tilde{\Delta}(e^u)}{e^{\frac{3}{2}u}} du \\ &= \operatorname{Re} [s(s+1) \int_1^{+\infty} \tilde{\Delta}(x) x^{-s-\epsilon-2} dx]_{s=\frac{1}{2}+ik} \end{aligned}$$

Using integration by parts couple of times, we get

$$\begin{aligned} s(s+1) \int_1^X \tilde{\Delta}(x) x^{-s-\epsilon-2} dx &= s\tilde{\Delta}(1) - \Delta(1) + \int_1^X x^{-s-\epsilon} d\Delta(x) + \epsilon \int_1^X x^{-s-\epsilon} \Delta(x) dx \\ &\quad - s\epsilon \int_1^X x^{-s-\epsilon-2} \tilde{\Delta}(x) dx - s x^{-s-\epsilon-1} \tilde{\Delta}(x) \end{aligned}$$

and

$$\int_1^X x^{-s-\epsilon} d\Delta(x) = \int_1^X x^{-s-\epsilon} d\psi(x) - \int_1^X x^{-s-\epsilon} dx = \sum_{n \leq X} \frac{\Lambda(n)}{n^{s+\epsilon}} - \frac{X^{1-s-\epsilon}}{1-s-\epsilon} + 1$$

By the formula 2, when $\operatorname{Re} s \geq \frac{1}{2}$ we get that

$$\int_1^{+\infty} x^{-s-\epsilon} d\Delta(x) = -\frac{\zeta'(s+\epsilon)}{\zeta(s+\epsilon)} + 1 \quad (43)$$

Therefore

$$\int_0^{+\infty} e^{-\epsilon u} \left[\left(\frac{3}{4} - k^2 \right) \cos(ku) + 2k \sin(ku) \right] \frac{\tilde{\Delta}(e^u)}{e^{\frac{3}{2}u}} du = \operatorname{Re} \left[\frac{\zeta'(s+\epsilon)}{\zeta(s+\epsilon)} \right]_{s=\frac{1}{2}+ik} + \varphi_\epsilon(k) \quad (44)$$

Where

$$\varphi_\epsilon(k) = \operatorname{Re} [-s\epsilon \int_1^\infty x^{-s-\epsilon-2} \tilde{\Delta}(x) dx + \epsilon \int_1^\infty x^{-s-\epsilon-1} \Delta(x) dx]_{s=\frac{1}{2}+ik}$$

For the simplicity, we denote LHS of 44 by $J_\epsilon(k)$. Choosing any test function $g(k) \in C_0^\infty(R^+)$, where $C_0^\infty(R^+)$ is the set of all smooth functions which

have compact supports on R^+ , let's compute following inner product, by 44

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \langle J_\epsilon(k), g(k) \rangle &= Re \left[\frac{\zeta'(s + \epsilon)}{\zeta(s + \epsilon)} \right] \Big|_{s=\frac{1}{2}+ik} + \langle \varphi_\epsilon(k), g(k) \rangle \\ &= \lim_{\epsilon \rightarrow 0} \langle Re \left[\frac{\zeta'(s + \epsilon)}{\zeta(s + \epsilon)} \right] \Big|_{s=\frac{1}{2}+ik}, g(k) \rangle + \lim_{\epsilon \rightarrow 0} \langle \varphi_\epsilon(k), g(k) \rangle \end{aligned} \quad (45)$$

and it's not difficult to verify that $\lim_{\epsilon \rightarrow 0} \langle \varphi_\epsilon(k), g(k) \rangle = 0$

Using integration by parts twice, we have that

$$\langle Re \left[\frac{\zeta'(s + \epsilon)}{\zeta(s + \epsilon)} \right] \Big|_{s=\frac{1}{2}+ik}, g(k) \rangle = \langle h(\frac{1}{2} + \epsilon + ik), g''(k) \rangle \quad (46)$$

Where $h(z) = \int_1^z \ln \zeta(s) ds$ and $Re z > \frac{1}{2}$, the integration path is the conventional contour from 1 to z

Since $h(\frac{1}{2} + \epsilon + ik) = O(k \log k)$ uniformly for any small ϵ and $\lim_{\epsilon \rightarrow 0} h(\frac{1}{2} + \epsilon + ik) = S_1(k)$

Where $S_1(k) = \int_0^k S(t) dt$

Finally, we have

$$\lim_{\epsilon \rightarrow 0} \langle J_\epsilon(k), g(k) \rangle = \langle S_1(k), g''(k) \rangle \quad (47)$$

Before ending this section, let's take look at the equation 39 again, we can rewrite this equation in term of integral equation as follows:

$$\int_0^\infty K_X(t, t') dN(t') = H_X(t) \quad (48)$$

Where

$$\begin{aligned} K_X(t) &= - \frac{(\frac{3}{4} - t^2)(\frac{3}{4} - t'^2) + 4tt' \sin(t' - t) \ln X}{(\frac{3}{4} - t'^2)^2 + 4t'^2} \frac{1}{t' - t} - \frac{(\frac{3}{4} - t^2)(\frac{3}{4} - t'^2) - 4tt' \sin(t' + t) \ln X}{(\frac{3}{4} - t'^2)^2 + 4t'^2} \frac{1}{t' + t} \\ &\quad + \frac{\frac{3}{2} + 2tt'}{(\frac{3}{4} - t'^2)^2 + 4t'^2} \cos(t' - t) \ln X + \frac{\frac{3}{2} - 2tt'}{(\frac{3}{4} - t'^2)^2 + 4t'^2} \cos(t' + t) \ln X \end{aligned}$$

and

$$H_X(t) = \int_1^X \frac{\tilde{\Delta}(x)}{x^{\frac{5}{2}}} [(\frac{3}{4} - t^2)\cos(t\ln x) + 2t\sin(t\ln x)]dx + \frac{\ln \pi}{2} - \frac{1}{2}Re\frac{\Gamma'}{\Gamma}(\frac{1}{2} + it)$$

Actually, above equation depends on the parameter X , for every fixed X , we get a non trivial integral equation of $N(t)$, so we obtain a family of integral equations, noticing that the integral kernel $K_X(t, t')$ is an explicit function, it's expected that exploring these integral equations will help us to understand RH further, besides we can consider similar results for L function which satisfies functional equation.

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